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The effect of initial conditions on the development of a free shear layer

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The distance between the separation point and the final approach to a fully developed turbulent mixing layer is found to be of the order of a thousand times the momentum-deficit thickness of the initial boundary layer, whether the latter be laminar or turbulent. There are correspondingly large shifts in the virtual origin of the mixing layer, resulting in spurious Reynolds-number effects which cause considerable difficulties in tests of model jets or blunt-based bodies, and which are probably responsible for the disagreements over the influence of Mach number on the development of free shear layers. These effects are explained.

1. Introduction

Since the ratio of Reynolds shear stress to viscous stress in a free turbulent flow is proportional to the Reynolds number (being about $0{\cdot}002U_1x/\nu$ in the case of a mixing layer or 'half jet'), the flow should be independent of the local Reynolds number when the latter is high. Therefore any effects of Reynolds number on such a flow must be exerted through the boundary layer at the separation point. In what follows we shall discuss these effects with particular reference to the mixing layer from a constant-area nozzle, but the effects in base flows and separation bubbles will be similar. In the case of the mixing layer the velocity profiles in the fully developed region take the self-preserving form $U/U_1 = f\{y/(x-x_0)\}$, where the origin of the co-ordinates is at the separation point (the nozzle lip), and $x = x_0$ is the virtual origin of the flow, which depends on the initial conditions and therefore on the Reynolds number. At first sight one would expect that the virtual origin would be upstream of the nozzle by a few times the thickness of the boundary layer at $x = 0$, and this is the prediction of the 'mixing-length' theories of the development of a mixing layer from a turbulent boundary layer, reviewed by Nash (1962). However, the virtual origin of the mixing layer developing from a turbulent boundary layer about 0·05 in. thick in a 2 in. diameter nozzle at speeds of the order of 300 ft./sec was about 150 momentum-deficit thicknesses *downstream* of the nozzle. When the boundary layer was laminar x_0 varied from $-200\delta_2$ to $+350\delta_2$ as the speed varied from 150 to 600 ft./sec. It is therefore clear that the effects of the initial conditions are considerably larger than has been generally appreciated in the past.

It will be shown below that, when the initial boundary layer is laminar, the shear stress in the transition region where the velocity fluctuations are still sinusoidal can rise to double the value in the fully developed turbulent flow, the actual peak value depending strongly on the Reynolds number: moreover, the decay of this excess shear stress and the subsequent establishment of the self-preserving turbulent flow is slow to take place. If the boundary layer is turbulent the increase of turbulent kinetic energy to the much higher level in the mixing layer absorbs a large proportion of the available energy production for a considerable distance downstream, so that self-preservation is again attained slowly.

Fundamental investigations of the transition process have been reported by Wille (1963*a*, *b*) and his collaborators, and by Sato (1956, 1959): Wille (1963*b*) also discusses the effect of the initial conditions on the fully developed flow. Kelly (1965) has studied a simplified theoretical model of the onset of non-linearity in the transition process. Chapman, Kuehn & Larson (1958) give examples of the large effect of transition in free shear layers in their study of step-induced separation, and the related subject of Reynolds-number effects on bubble separation is discussed by Tani (1964).

This paper is intended to summarize the effect of initial conditions because their magnitude, particularly in the case of a turbulent initial boundary layer, does not seem to be generally appreciated by workers studying free shear layers. In addition, the phenomena that occur are intrinsically interesting and worthy of further theoretical study.

2. Apparatus

A 2 in. diameter jet (Bradshaw, Ferriss & Johnson 1964) was used for all the measurements, which were confined to the first few diameters downstream of the nozzle where the mixing layer is quasi-plane. Parallel sections of various lengths could be screwed on to the end of the nozzle to alter the initial boundary-layer thickness: when the layer was laminar, the momentum thickness was about

$$0{\cdot}00086\sqrt{[(L+2{\cdot}7)/M]}\ \text{in.},$$

where L is the length of the nozzle extension in inches and M is the Mach number. A carefully machined trip ring could be inserted to precipitate transition: the momentum thickness of the turbulent boundary layer in the $1\frac{1}{4}$ in. long nozzle extension was 0·006 in. at $M = 0{\cdot}3$, or double the thickness of the untripped layer in the same nozzle at the same speed.

A twin Pitot probe was used for measurements of transverse velocity gradient, $\frac{1}{2}\rho U_1^2/(\partial P/\partial y)$ at $y/r_0 = 0$ being taken as a simple comparative measure of shear-layer thickness for purposes of determining the virtual origin of the fully developed flow. The probe consisted of two flattened Pitot tubes 0·045 in. wide and 0·015 in. high mounted 0·03 in. apart. The difference Δp between the readings of the two tubes was thus $l\,\partial P/\partial y$, where l is the 'effective' distance between the tubes. The probe was calibrated by traversing it right across the shear layer, and using the result

$$\int_{-\infty}^{\infty} \Delta p.dy = l\int_{-\infty}^{\infty} (\partial P/\partial y)\,dy = l\,.\,\tfrac{1}{2}\rho U_1^2.$$

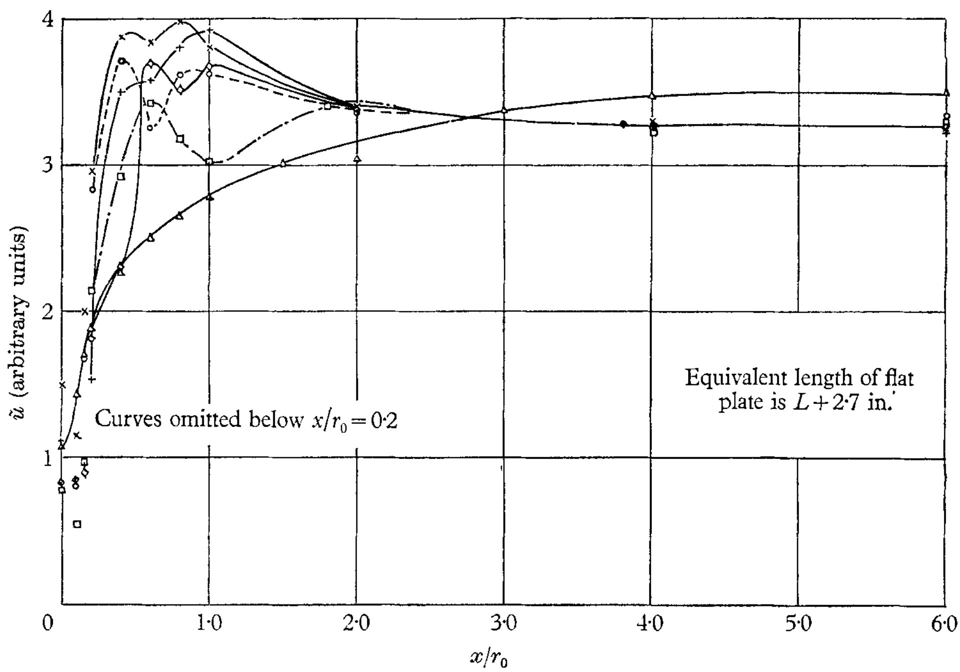
The distance l varied slightly with axial position, rising by 10 % between $x/r_0 = 2$ and $x/r_0 = 8$, as the scale of the turbulent eddies changed. This is merely a general reflexion upon the use of Pitot and static tubes in highly turbulent flows and not a condemnation of the twin Pitot probe as such, but in view of the variations in l the measurements of the shear layer 'thickness' are presented in arbitrary units and for one axial position only.

3. The transition process

The free shear layer emerging from a jet nozzle becomes unstable to infinitesimal disturbances at almost any Reynolds number as soon as a point of inflexion appears in the velocity profile—that is, immediately after the jet leaves the nozzle. Therefore the frequency of the primary disturbance wave is expected to depend only on the velocity gradient in the nozzle boundary layer near the surface, $(\partial U/\partial y)_0$. For a Blasius boundary-layer profile, the present experiments and the work of Sato (1956, 1959) show that the frequency is about $\omega\delta_2/U_1 = 0{\cdot}11$ or $\omega/(\partial U/\partial y)_0 = 0{\cdot}47$ for $100 < U_1\delta_2/\nu < 500$ at least. The approximate constancy of the dimensionless frequency over a range either side of the critical Reynolds number of the Blasius flow, 160, is an indication that the disturbances that amplify arise in the free shear layer and not in the boundary layer upstream, but if the Reynolds number is high enough transition will of course occur upstream of the nozzle exit. Wille observed that the natural frequency in the shear layer issuing from a rather abrupt nozzle, with a free-stream velocity at the lip about 1·06 times that on the axis, was only half the value quoted above: this is an indication that the pressure gradient near separation, which determines the profile shape, may have a large effect on the frequency, particularly when separation is induced by a pressure gradient and not by a surface discontinuity. If $\omega/(\partial U/\partial y)_0 = 0{\cdot}47$, Wille's result implies that his exit boundary layer was retarded, with $\delta_1/\delta_2 \simeq 2{\cdot}9$.

In a circular jet, the disturbances take the form of 'vortex rings' (see plate 1 of Bradshaw *et al.* 1964) which are in phase right round the circumference, probably because the induced velocity field of a vortex element initially extending over part of the circumference produces a slight pulsation in the velocity of the jet as a whole, thus synchronizing disturbances right round the circumference: certainly the correlation is unaffected by a diametral splitter plate. In a 'two-dimensional' jet the vortex lines on either side of the jet are correlated but apparently either in phase or in antiphase. (The use of the terms 'vortex ring' and 'vortex line' should strictly be reserved for concentrations of vorticity. The fact that filament lines roll up is quite compatible with a sinusoidal variation of vorticity such as occurs in the present case: the filament lines roll up at the points where $\partial U/\partial y$ is a maximum or a minimum. However, Pierce (1964) has shown that quite accurate predictions of the behaviour of a curved free shear layer can be made by supposing discrete vortices to be placed at these points.) It is noticeable that the frequency and amplitude of the oscillations at a given point are much more nearly constant at speeds of two or three hundred feet per second than at lower speeds. At these speeds a discrete-frequency whistle can sometimes be heard, and at lower speeds than this a whistle can be produced by inserting a razor blade, edge on, into the jet, and this also stabilizes the frequency, so it appears that the frequency of the initial disturbances can be controlled by the sound emitted from the more intense vortex rings farther downstream, or from the excitation of the nozzle lip by the irrotational field of those vortex rings (Ffowcs Williams & Gordon 1965): this phenomenon closely resembles shock-cell noise or 'screeching' of choked jets. Since the sound radiation is a function of Mach number, we are faced with the possibility of slight Mach-number effects on the transition process at speeds low enough for the direct effects of compressibility to be utterly negligible.

The 'vortex ring' disturbances grow to a very high intensity although the velocity fluctuation remains roughly sinusoidal. The u and v component r.m.s. intensities and the Reynolds shear stress are shown in figure 1 for different initial



(*a*)

FIGURE 1. For legend see facing page.

boundary-layer thicknesses: the u-component results were obtained with a single-wire probe, the u-component results from the X-probe runs being similar but more scattered. The results are presented in arbitrary units: measurements reported by Bradshaw *et al.* show that at $x/r_0 = 4$, $y/r_0 = 0$ the values are $\tilde{u}/U_1 \simeq 0{\cdot}13$, $\tilde{v}/U_1 \simeq 0{\cdot}13$, $\overline{uv}/U_1^2 \simeq 0{\cdot}01$. The v-component r.m.s. intensity (figure 1 (*b*)) reaches almost twice its 'fully developed' value, implying a nearly sinusoidal fluctuation of half-amplitude more than 0·35 of the free-stream velocity. The Reynolds shear stress (figure 1 (*c*)) reaches a very sharp maximum at 155 ± 8 momentum thicknesses from the nozzle: the v-component peak is just significantly downstream of this point. This is about the point where the disturbance frequency suddenly halves and, according to Wille, confluence of vortex rings occurs. This is presumably the process which appears in flow visualization pictures as an apparent contraction of the jet at the end of the primary vortex stage (see also the oscilloscope traces and frequency spectra of Sato 1959, which show that the same phenomenon occurs in two-dimensional flow). One would

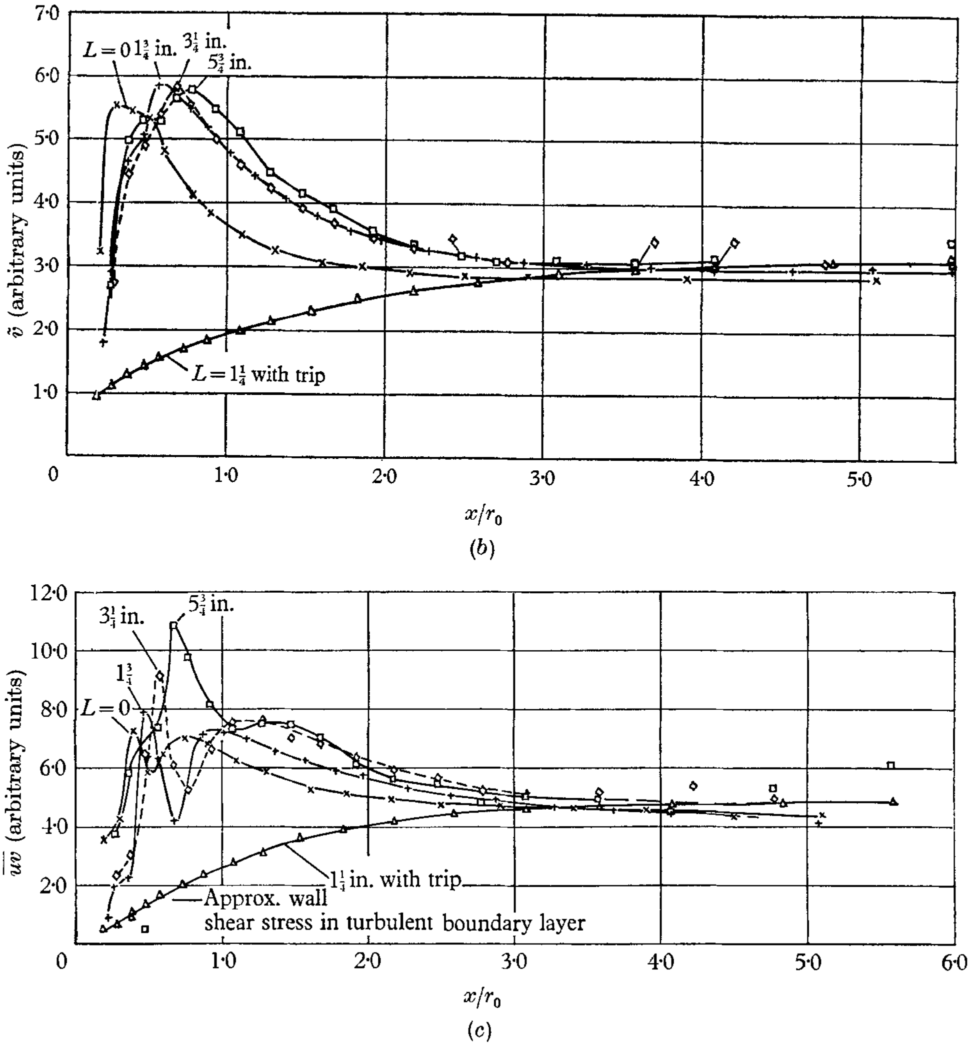


FIGURE 1. (*a*) Variation of u-component intensity with x for different thicknesses of exit boundary layer. $y/r_0 = 0$, $M = 0{\cdot}3$, $r_0 = 1$ in. —×, $L = 0$; ----, $L = 1\frac{1}{4}$ in.; —△, $L = 1\frac{1}{4}$ in. with trip; —+, $L = 1\frac{3}{4}$ in.; —◇, $L = 3\frac{1}{4}$ in.; ———□, $L = 5\frac{3}{4}$ in. (*b*) Variation of v-component intensity with x for different thicknesses of exit boundary layer. $y/r_0 = -0{\cdot}015$, $M = 0{\cdot}3$, $r_0 = 1$ in. (*c*) Variation of Reynolds shear stress with x for different thicknesses of exit boundary layer. $y/r_0 = -0{\cdot}015$, $M = 0{\cdot}3$, $r_0 = 1$ in.

expect the process, which seems to involve the passage of one vortex ring inside another, to result in strong transfer of u-component momentum in the y-direction (viz. a high shear stress). In the axisymmetric case the consequent change of perimeter of a vortex ring will lead to a change in vorticity, so that the ratio of the thickness of the shear layer to the radius of the ring, or δ_2/r_0, may be a relevant parameter even when it is quite small. It is most interesting to note that the maximum value of the shear stress increases with boundary-layer thickness, or, more precisely, with $U_1\delta_2/\nu$. Downstream of the maximum, the shear stress falls almost as rapidly as it rose, showing that the confluence of vortex rings is accomplished quite quickly. The shear stress reaches a second maximum, and the u-component intensity sometimes does likewise although v does not: the distance between the two maxima of $\overline{uv}$ rises from about $145\delta_2$ at $U_1\delta_2/\nu = 450$ to $160\delta_2$ at $U_1\delta_2/\nu = 800$. This second maximum presumably marks the establishment of the shear-producing part of the turbulence spectrum: the shear stress is higher than the fully developed value because the smaller-scale turbulence, which drains energy from the shear-producing eddies by a sort of 'eddy viscosity' mechanism, has not yet been established. It appears from flow-visualization pictures that breakdown to turbulence occurs via the longitudinal vortices predicted by Benney (1961).

Transition in a free-mixing layer with a Blasius boundary-layer profile at separation is thus described by a length scale δ_2 and a velocity scale U_1, with parameters $U_1\delta_2/\nu$, M and δ_2/r_0. Additional variables are introduced in the case of separation from bluff bodies because of the effects of pressure gradient on the boundary layer or the free shear layer or both. It is doubtful whether empirical formulae describing the behaviour of the transition region in terms of all these parameters could be derived without a great deal of work, although a more thorough theoretical and experimental study of the later stages of transition would be of considerable fundamental interest. The only theoretical work on the halving of the disturbance frequency and the associated effects is Kelly's (1965) study of a parallel temporally amplifying flow which represents the qualitative features of the real spatially amplifying flow very well: there is as yet no quantitative theory with which the intensity measurements could be compared. Since the immediate practical interest in the problem is confined to the establishment of a fully developed turbulent shear layer for model tests, it will be sufficient to obtain approximate formulae for the total distance required to attain full development. This question, and the behaviour of a free shear layer developing from a *turbulent* boundary layer, will be discussed in the next section.

4. The approach to full development

When the free-stream speed is doubled, the initial boundary layer remaining laminar, the distance from the nozzle to the point of maximum u-component intensity decreases by a factor of about $1/\sqrt{2}$, confirming Wille's finding that the distance is a multiple of the boundary-layer thickness and effectively independent of Reynolds number. The subsequent distance to the attainment of self-preservation (constancy of turbulent intensity) seems to decrease by a factor nearer $\frac{1}{2}$ in the present experiments. This, together with the observation from figures 1(a) to 1(c) that this distance increases only slowly with increase in boundary-layer thickness, suggests that the Reynolds number based on this distance may be roughly constant, although there must be some dependence on δ_2. The total distance to attain full development from a laminar boundary layer at exit can be expressed approximately as the sum of the distance to vortex-ring breakdown, $160\delta_2$, and the subsequent distance to full development, say $4 \times 10^5 \nu/U_1$ for $500 < U_1\delta_2/\nu < 1000$: in view of the uncertainties and the other parameters that may enter, it is probably good enough to quote a total distance of $7 \times 10^5 \nu/U_1$ for the same Reynolds-number range. The extensive turbulence measurements of Bradshaw *et al.* (1964) were made at about this distance from the nozzle, following a briefer investigation of the effect of exit conditions than that reported here.

When the initial boundary layer is turbulent, the fluctuation intensities and the shear stress increase monotonically, and very slowly, from the exit. All three significantly overshoot the fully developed value and do not seem to decrease

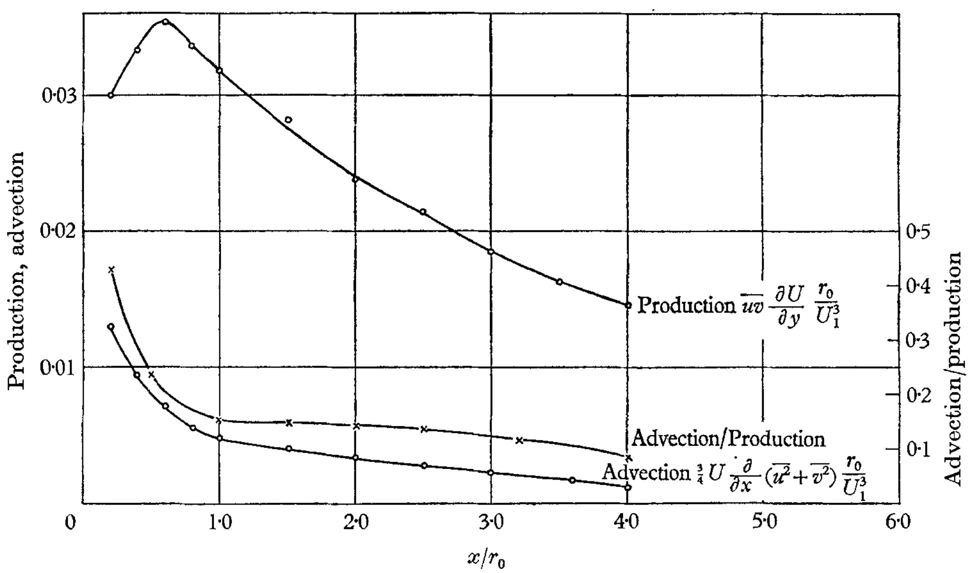


FIGURE 2. Production and advection of turbulent energy: turbulent exit boundary layer ($L = 1\frac{1}{4}$ in. with trip). $y/r_0 = 0$.

again within the quasi-plane region of the shear layer. It is clear that turbulent initial boundary layers are to be avoided when one is trying to set up a self-preserving mixing layer. In a first, unpublished version of this report it was implied that these effects were confined to model scale, because the experiments indicated that the Reynolds number based on the distance to full development was roughly constant as in the case of a laminar initial boundary layer. However, this is difficult to reconcile with the principle of Reynolds-number independence mentioned in the introduction and I am now less confident of its reality. In figure 2 the approximate rate of production of turbulent energy is compared with the major part of the advection term: it is seen that the advection is a large fraction of the production for small x and remains at 10–15 % of the production until $x/r_0 = 4$ (it is of course zero at $y = 0$ in the fully developed mixing layer). Since the advection cannot be as large as the production it is implied that the distance to full development, measured as a multiple of the initial boundary-layer thickness, cannot be much smaller at full scale than at model scale. Departures from self-preservation in full-scale mixing layers, particularly in base flows and at the exit from long jet pipes, may therefore be significant at distances from the separation point of $1000\delta_2$. This phenomenon is an extreme example of the effect of past history on the development of a turbulent flow. The 'mixing-length' theories reviewed by Nash (1962) are not capable of representing this effect: the mixing-length approach also fails, although not so spectacularly, in the case of the (attached) turbulent boundary layer.

We now have approximate formulae for the distance required for a free mixing layer to develop from an initial boundary layer: it remains to study the effect of the initial conditions on the virtual origin of the fully developed flow.

5. The behaviour of the virtual origin

The twin Pitot probe was used to measure the 'thickness' $\frac{1}{2}\rho U_1^2/(\partial P/\partial y)_{y=0}$ of the shear layer at $x/r_0 = 4$ over a range of exit speeds for different thicknesses of initial boundary layer. The results are shown in figure 3, in which an approximate scale of x_0, the downstream distance from the nozzle exit to the virtual origin, is shown by the side of the 'thickness' scale. We first note the extremely large variations in thickness over the speed range for a given nozzle length, even in the case of the shortest nozzle of all ($L = 0$). The worst of the variation occurs below a speed of about 300 ft./sec, when the mixing layer at $x/r_0 = 4$ is not necessarily fully developed (compare figure 1): the $5\frac{3}{4}$ in. long nozzle shows abrupt variations near the speed at which transition first occurs inside the nozzle and the $3\frac{1}{4}$ in. nozzle behaves erratically over the whole speed range. The trend with increasing nozzle length is partly obscured by these eccentricities but it is clear that, at a given exit speed, the virtual origin moves upstream as the thickness of the (laminar) boundary layer increases, and then moves quickly downstream when transition occurs within the nozzle. We particularly note that when the initial boundary layer is turbulent the virtual origin is nearly an inch (say 150 momentum thicknesses) *downstream* of the nozzle except at the lowest Mach numbers when the results were influenced by the increasing thickness of the laminar boundary layer upstream of the trip. The effect of the finite width of the shear layer at the separation point is completely swamped by the slow approach to the fully-developed value of shear stress.

Empirical formulae for the position of the virtual origin, in the form $x_0/\delta_2 = f(U_1\delta_2/\nu)$, would not be very reliable or very useful in practice: in an actual experiment it would be best to measure x_0 directly, having chosen the dimensions of the test rig in accordance with the formulae giving the approximate distance to full development. We merely note that the shift in virtual origin may be several hundred times the momentum thickness of the initial boundary layer and is therefore unlikely to be negligible in model experiments. In the experiments of Davies, Barratt & Fisher (1963) in a 1 in. diameter jet the virtual origin was about 0·4 in. upstream of the orifice at Mach numbers of 0·2 to 0·4. In the experiments of Bradshaw *et al.* (1964) in a 2 in. jet at $M = 0{\cdot}3$ (using the same test rig as in the present work, with the $1\frac{1}{4}$ in. long nozzle extension) the virtual origin was only 0·25 in. upstream of the nozzle: it was only during the course of the present work that it was realized that this was a remarkable coincidence and that very much larger shifts, in either direction, are to be expected in practice.

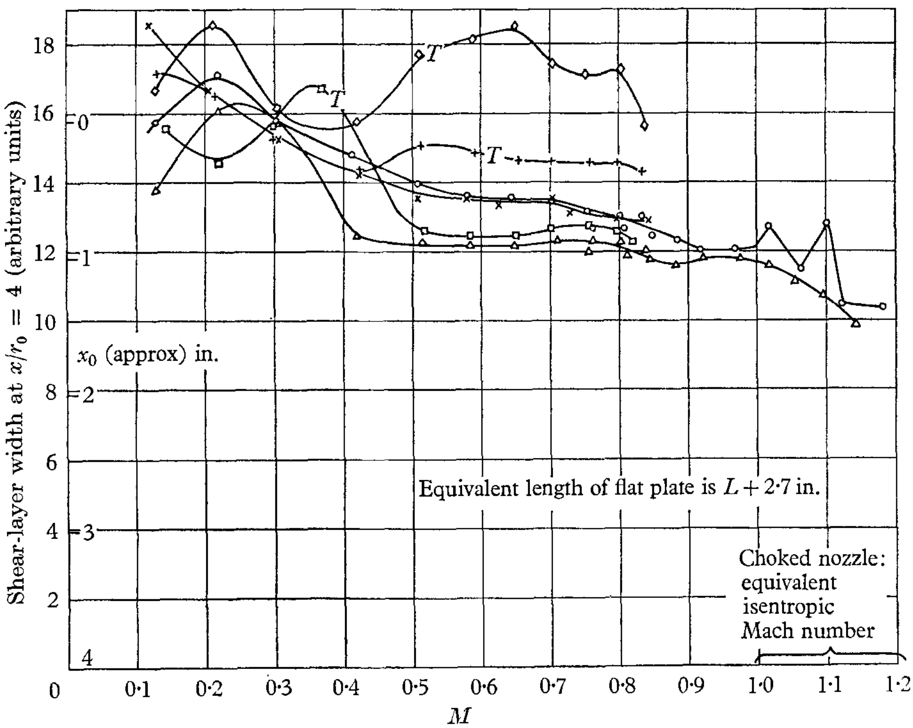


FIGURE 3. Variation of shear-layer width at $x/r_0 = 4$ with Mach number for different thicknesses of exit boundary layer. $\times$, $L = 0$; $\bigcirc$, $L = 1\frac{1}{4}$ in.; $\triangle$, $L = 1\frac{1}{4}$ in. with trip; $+$, $L = 1\frac{3}{4}$ in.; $\diamond$, $L = 3\frac{1}{4}$ in.; $\square$, $L = 5\frac{3}{4}$ in.; T indicates transition.

6. Measurements of noise emission

If the transition region of the free shear layer were noiseless and the effect of initial conditions could be represented entirely by a shift of virtual origin, the noise emission would be unaffected. In fact, the transition region or the non-self-preserving part of the turbulent flow produce a good deal of noise, so that remarks made in previous sections about the effect of initial conditions on the shear layer development apply also to the noise emission: the effect of tripping the boundary layer, for instance, is to reduce the high-frequency content of the noise by an amount obvious to the unaided ear.

Measurements of the sound pressure level with various thicknesses of initial boundary layer are plotted in figure 4 as a 'noise emission coefficient'

$$\mathrm{SPL\,dB} - 80\log_{10} Ma/a_0,$$

so that if Lighthill's (1963) U^8 law were obeyed there would be no variation with Mach number. It is seen that generally the sound pressure level decreases as the boundary-layer thickness increases. Mollo-Christensen, Kolpin & Martuccelli

(1964) also discuss the noticeable effect of exit conditions on the noise radiated by a jet and Mollo-Christensen (1963) describes erratic variations of near-field pressure fluctuations which are in fact attributable to transition from a laminar to a turbulent exit boundary layer.

The most interesting feature of the present results is that the noise-emission coefficient rises rapidly as the Mach number is decreased below 0·5 (the jet exhausted into a non-reverberant room and the noise from the air-supply pipe

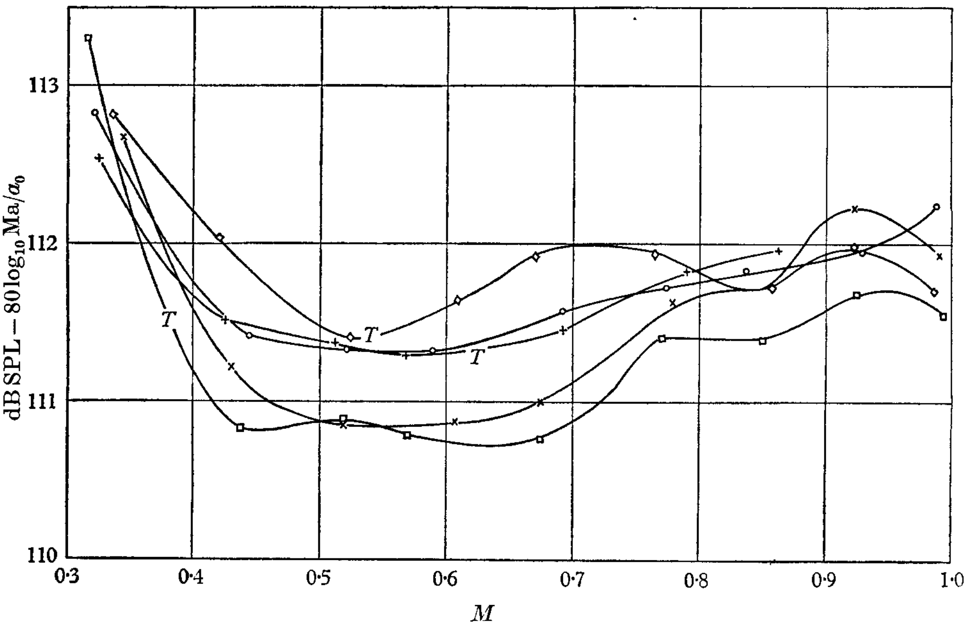


FIGURE 4. Variation of 'noise-emission coefficient' with Mach number for different thicknesses of exit boundary layer. ×, $L = 0$; ○, $L = 1\frac{1}{4}$ in.; +, $L = 1\frac{3}{4}$ in.; ◇, $L = 3\frac{1}{4}$ in.; □, $L = 5\frac{3}{4}$ in.; T indicates transition.

line, measured with the contraction removed, was considerably less than the jet noise even at very low Mach numbers of the order of 0·2). It seems most probable that this is caused by the lengthening of the transition region as the speed is reduced, and is therefore a Reynolds-number rather than a Mach-number effect. A recent paper by Ffowcs Williams & Gordon (1965) discusses several sources of noise emission in low-speed flows, including simple-source emission, caused by unsteadiness of the exhaust mass flow, and dipole excitation of the nozzle lip. The former is unlikely to be important in the very steady flow of the 2 in. jet rig and the latter, while it probably accounts for the discrete frequency whistle sometimes heard, is not likely to be the explanation of the broad-band high-frequency noise from the transition region: in view of the high fluctuation intensities observed it is likely that the quadrupole emission (which is not confined to the airflow frequencies measured by a fixed observer) is quite large enough to be entirely responsible.

The presence of this Reynolds-number effect on jet noise implies that experiments with model jets at low Mach numbers should be undertaken with caution and viewed with suspicion. In particular, verification of theories of the simple-source and dipole emission discussed by Ffowcs Williams & Gordon may be very difficult.

7. Mach-number effects on shear layer development

Judging by the results of figure 3 there are no appreciable true Mach-number effects up to $M = 0{\cdot}85$ at least. This conclusion is in conflict with the deductions which have been made from the work of Laurence (1956) and Lassiter (1957), both of whom found significant changes in mean flow and turbulence level in the subsonic-speed range. Laurence referred to 'Mach/Reynolds number effects' and the words 'Mach number' do not appear in Lassiter's report at all, but subsequent commentators have tended to assume that Mach number was the dominant variable. Lassiter's boundary layer was almost certainly turbulent and there are indications that Laurence's was transitional or turbulent: in view of the present results we may be fairly certain that the 'Mach-number effects' were really the effect of the exit boundary layer.

It is likely that the considerable scatter among the various results for spreading rate of supersonic jets can be partly attributed to exit boundary-layer effects in the very small nozzles used by some workers. It is salutary to note that the exit boundary-layer thickness increases with Mach number simply because the length of a convergent–divergent nozzle increases with Mach number. It may also be remarked that erratic results may be obtained, even in subsonic flow, with nozzles that do not give an exactly parallel flow at exit: Wille (1963*b*, figure 14) gives examples of this which are not entirely attributable to the influence of the profile shape of the initial boundary layer on the transition process. Maydew & Reed (1963) discuss the measurements of a large number of workers and give details of their own experiments in a 3 in. jet, which appear to be among the most reliable because the jet was large enough to give a small ratio of exit boundary-layer thickness to nozzle diameter and because the shift of virtual origin was allowed for in the results. They conclude that compressibility effects are negligible at Mach numbers less than unity.

I am grateful to Dr J. T. Stuart for several useful discussions of the transition process. The experimental work was carried out by G. K. Knight and P. C. Chatwin (Vacation Student, summer 1963).

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REFERENCES

Benney, D. J. 1961 *J. Fluid Mech.* **10**, 209.

Bradshaw, P., Ferriss, D. H. & Johnson, R. F. 1964 *J. Fluid Mech.* **19**, 591.

Chapman, D. R., Kuehn, D. M. & Larson, H. K. 1958 *Nat. Adv. Comm. Aero. (Wash.) Rep.* no. 1356.

Davies, P. O. A. L., Barratt, M. J. & Fisher, M. J. 1963 *J. Fluid Mech.* **15**, 337.

Ffowcs Williams, J. E. & Gordon, C. G. 1965 *J. Amer. Inst. Aeron. & Astron.* **3**, 791.

Kelly, R. E. 1965 *Nat. Phys. Lab. Aero Rep.* no. 1161.

Lassiter, L. W. 1957 *J. Appl. Mech.* **24**, 349.

Laurence, J. C. 1956 *Nat. Adv. Comm. Aero.* (*Wash.*) *Rep.* no. 1292.

Lighthill, M. J. 1963 *J. Amer. Inst. Aeron. & Astron.* **1**, 1507.

Maydew, R. C. & Reed, J. F. 1963 *J. Amer. Inst. Aeron. & Astron.* **1**, 1443.

Mollo-Christensen, E. 1963 *Mass. Inst. Tech. Rep.* no. ASRL-1006.

Mollo-Christensen, E., Kolpin, M. A. & Martuccelli, J. R. 1964 *J. Fluid Mech.* **18**, 285.

Nash, J. F. 1962 *Aero. Res. Counc.* (*Lond.*) *Curr. Pap.* no. 682.

Pierce, D. 1964 *Roy. Aircr. Estab. Tech. Note Aero.* no. 2970.

Sato, H. 1956 *J. Phys. Soc.* (*Japan*), **11**, 702.

Sato, H. 1959 *J. Phys. Soc.* (*Japan*), **14**, 1797.

Tani, I. 1964 *Prog. Aero. Sci.* **5**, 70.

Wille, R. 1963*a* *Berlin Tech. Univ. Hermann Fottinger Inst. für Strömungstech.* AF 61(052)-412.

Wille, R. 1963*b* *Z. Flugwiss.* **11**, 222.